Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 4, 1996

A CLASS OF SHEAR FLOWS OF A VISCOUS COMPRESSIBLE FLUID

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UDC 517.958

Introduction. A variety of flows of a viscous compressible fluid are described by one-dimensional equations. Antontsev et al. [1] have studied mathematical problems of flows with plane waves and rectilinear trajectories of fluid particles. This paper is concerned with a more general class of motion with plane waves without conditions on fluid-particle trajectories. An example of such motion is flow between two parallel planes when each liquid plane parallel to the boundary planes moves irrotationally as a solid body; in this case, all fluid particles have the same velocity. It is clear that in this motion the trajectories of all particles are the same but not necessarily rectilinear.

To these motions correspond solutions of three-dimensional equations of motion that are invariant with respect to the group of translations over variables changing in the bounding planes. For simplicity, we consider these planes horizontal.

The invariance restrictions imply that only derivatives with respect to time and to the spatial vertical variable remain in the equations. All the functions entering into the equations are independent of the spatial horizontal coordinates. In this case, the horizontal velocity components satisfy parabolic equations that do not contain pressure. And for the vertical velocity component we have a system in which the horizontal components are incorporated via an energy equation.

Thus, the invariance condition virtually reduces the initial system of three-dimensional equations to a one-dimensional system which takes into account translations of liquid layers relative to each other in contrast to the equations for rectilinear trajectories [1].

This paper proves that this problem is uniquely solvable when the pressure depends only on the density and the energy–temperature relation is identical to that for a perfect gas. Thus, the possibility for irrotational movement of the liquid layers is substantiated. On the other hand, it is established that motion of the liquid layers with an additional degree of freedom, i.e., when the material planes can rotate as a solid body with an angular velocity which is directed vertically and different for each plane, is impossible.

As an example of the use of invariant solutions, we consider the role of shear flow in the formation of a temperature regime for a compressible viscous fluid layer. The asymptotic behavior of temperature with time is investigated for the case where the layer is heat-insulated and the upper bounding plane moves uniformly at a constant velocity and at a constant distance from the fixed lower bounding plane. The proof is given that the temperature rises in a linear fashion.

Remark 1. For perfect isothermal gases, the invariance restriction assumes that the trajectories are rectilinear in some reference system. This follows from the fact that the horizontal components of the particle velocity do not vary with time [2], and, hence, it is possible to move to another inertial reference system in which one cannot observe horizontal motion.

Formulation of the Problem and the Main Result. Flow of a compressible viscous fluid layer ($0 < x < 1$) between two planes, one of which (upper) moves irrotationally at a distance $h = 1$ from the fixed lower plane, is investigated in a Cartesian coordinate system. The gravity field is directed downward along the x axis.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 4, pp. 50–56, July–August, 1996. Original article submitted February 21, 1995; revision submitted May 24, 1995.

It is assumed that the stress tensor **P** and the strain rate tensor **D** are in the same rheological relation as for the Navier–Stokes model [1]:

$$\mathbf{P} = (-p + \lambda \text{ div } \mathrm{v})\mathbf{I} + 2\mu\mathbf{D},$$

where λ and μ are the viscosity coefficients, p is the pressure, **v** is the velocity vector, and **I** is the unit tensor. Let u, v, and w be the fluid velocity vector components that are directed along the x, y, and z axes, respectively; let ρ be the density, θ the temperature, and U the specific internal energy. Under the assumption that all these flow parameters depend only on the variable x and time t, the laws of conservation of momentum, mass, and energy for a viscous compressible fluid are of the form [1]

$$\rho(u_t + uu_x) = -p_x + \nu u_{xx} - \rho g; \tag{1}$$

$$\rho(v_t + uv_x) = \mu v_{xx}; \tag{2}$$

$$\rho(w_t + uw_x) = \mu w_{xx}; \tag{3}$$

$$\rho_t + (\rho u)_x = 0; \tag{4}$$

$$\rho(U_t + uU_x) = æ\theta_{xx} - pu_x + \nu u_x^2 + \mu(v_x^2 + w_x^2). \tag{5}$$

Here $\nu = \lambda + 2\mu$; æ is the heat conductivity; and g is the acceleration of gravity ($g \geqslant 0$). These parameters are considered positive constants. System (1)–(5) can be completed by the following equations of state:

$$p = \varphi(\tau), \qquad U = c_V\theta,$$

where $\tau = \rho^{-1}$ is the specific volume, c_V is a positive constant, and $\varphi(\tau)$ is a function that is specified on the real semiaxis $R^+ = \{r : r > 0\}$. In particular, for dense gases of the Tait type we have $\varphi(\tau) = A + B\tau^{-\gamma}$ (A and B are constants [2]). Note that the formulated model does not allow for thermodynamic equilibrium [3]. This means that, whatever the dependence of the entropy s on θ and τ, the thermodynamic identity $dU = \theta ds - pd\tau$ is generally incompatible with system (1)–(5) for the given equations of state.

It is assumed that the layer is heat-insulated and slipping in the bounding planes is absent. Let 0, V, and W be the components of the displacement velocity of the upper plane. Then, the equalities

$$u = v = w = \theta_x = 0 \quad \text{for} \quad x = 0; \qquad u = \theta_x = 0, \quad v = V(t), \quad w = W(t) \quad \text{for} \quad x = 1 \tag{6}$$

correspond to our assumptions.

The functions u_0, v_0, w_0, ρ_0, and θ_0 which specify the initial conditions

$$(u, v, w, \rho, \theta) = (u_0, v_0, w_0, \rho_0, \theta_0) \qquad \text{for} \qquad t = 0 \tag{7}$$

are considered dependent only on x.

This type of flow can also be described in Lagrangian variables. Let $\int_0^1 \rho_0(x)dx = 1$. If we introduce the mass Lagrange variable $x := \int_0^x \rho(\xi, t)d\xi$, then, in the new coordinates, the layer has a unit width as before and Eqs. (1)–(5) take the form [1]

$$u_t = \nu(\rho u_x)_x - p_x - g, \quad \tau_t = u_x, \quad \tau = \rho^{-1}; \tag{8}$$

$$v_t = \mu(\rho v_x)_x, \quad w_t = \mu(\rho w_x)_x; \tag{9}$$

$$c_V\theta_t = æ(\rho\theta_x)_x - pu_x + \mu\rho(v_x^2 + w_x^2) + \nu\rho u_x^2. \tag{10}$$

Boundary conditions (6) are thereby unchanged.

The movement of the upper plane acts as an external heat source, because of internal friction. The main goal of this paper is to obtain the asymptotic behavior of the temperature with time for the case where the upper plane moves at constant velocity V. More precisely, this paper seeks to prove that $\theta \to (\lambda/c_V)V^2t + E + \beta(x)$ as $t \to \infty$, where E is a constant.

Well-Posedness. The structure of Eqs. (8)–(10) is such that they can be solved sequentially. First, the functions u and τ are found from Eqs. (8), and, then, the functions v and w from system (9). After that the function θ is found from Eq. (10).

The initial boundary-value problem with conditions

$$u\Big|_{x=0,x=1} = 0, \qquad (u,\tau)\Big|_{t=0} = (u_0, \tau_0) \tag{11}$$

for system (8) has been studied in many papers. A detailed bibliography can be found in [1]. Let us present one result from [4] (it is valid, although it is obtained for $g = 0$). If the conditions

$$\begin{gathered} p(\tau) > 0, \quad p(1) = 1, \quad p'(\tau) < 0, \quad p(\tau) \in C^1(0,\infty), \quad \int_0^1 \tau_0(x)dx = 1, \\ 0 < m^{-1} \leqslant \tau_0 \leqslant m < \infty, \quad u_0(x), \tau_0(x) \in W_2^1(\Omega), \quad \Omega = \{x : 0 < x < 1\} \end{gathered} \tag{12}$$

are satisfied, there is a unique solution u, τ of problem (8) and (11) in an arbitrary time interval $[0,T]$. In this case, we have

$$\begin{gathered} u \in L_\infty\Big(0,T; \overset{\circ}{W}{}_2^1(\Omega)\Big) \bigcap L_2\Big(0,T; W_2^2(\Omega)\Big), \quad u_t, \tau_{xt} \in L_2\Big(0,T; L_2(\Omega)\Big), \\ \tau, \tau^{-1} \in L_\infty\Big(0,T; W_2^1(\Omega)\Big), \quad \tau > 0. \end{gathered} \tag{13}$$

Let v_0, w_0, $\theta_0 \in W_2^1(\Omega)$, and $v_0 - xV$, w_0 vanish when $x \in \partial\Omega$. As for parabolic equations (9), the properties of the function τ allow us to apply the well-known results [5] which guarantee the existence of the functions v and w of the same smoothness as that of u. With this smoothness of the functions u, τ, v, and w, Eq. (10) can be considered linear parabolic for θ. According to [5], this equation is uniquely solvable, and θ has the same smoothness as u. The latter fact can also be established using the approach in [1] for a polytropic gas.

Stabilization. The behavior of the solution of problem (8) and (11) with $t \to \infty$ has been studied in many papers. Kazhikhov et al. [4, 6] proved stabilization in the absence of external forces. Fluid flow in the field of external mass forces has been investigated in [7, 8]. It appeared that the limit stationary regime is not nondegenerate for all gases. In terms of equations of state and the external-force field, necessary and sufficient conditions were obtained when the stationary regime is not degenerate, i.e., the density does not vanish. Kolmogorov and Fomin [8] proved stabilization provided that the stationary regime is not degenerate.

The existence of a nondegenerate stationary regime $\tau = \tau_s$ and $u = 0$ is equivalent to the validity of the relations

$$p_{sx} = -g, \qquad \int_0^1 \tau_s(x)dx = 1, \qquad \tau_s > 0. \tag{14}$$

Let us perform a brief analysis of the density in a stationary state for problem (8) and (11) in the case where $p(\tau) = \tau^{-\gamma}$, $\gamma \geqslant 1$. The case of $0 < \gamma < 1$ has been studied in [7]. It is obvious that the analysis is reduced to finding a constant d such that

$$d > g, \qquad 1 = \int_0^1 (d - gx)^{-1/\gamma} dx. \tag{15}$$

The constant d, which is equal to $g(1 - \exp(-g))^{-1}$, satisfies these relations for $\gamma = 1$. For $\gamma > 1$, relations (15) are solvable with respect to d if and only if $g < (\gamma/(\gamma-1))^\gamma \equiv g^*$ (this condition was obtained in [8]). Indeed, if d is sought for in the form $d = qg$, from (15) follows an equation for q

$$g^r(\chi(q) - rg^{1-r}) = 0 \qquad (\chi(q) = q^r - (q-1)^r, \qquad r = 1 - 1/\gamma). \tag{16}$$

Since $\chi(q) \leqslant 1$ for $q \geqslant 1$ and $\chi(q) \to 0$ as $q \to \infty$, Eq. (16) is unsolvable for $g > g^*$ and uniquely solvable for

$g < g^*$. When $g = g^*$, the solution is $q = 1$, which is incompatible with the inequality from (15).

Let, along with (12), the function $p(\tau)$ satisfy the conditions below. For a positive constant Π, we have

$$\tau p(\tau) \leqslant \Pi(\Phi(\tau) + 1) \quad \text{for} \quad 0 < \tau \leqslant 1 \quad \text{and} \quad \lim_{\tau \to 0+} p(\tau) = \infty \quad \Big(\Phi(\tau) = \int_\tau^1 p(s)ds\Big). \tag{17}$$

In particular, these conditions are satisfied for $p(\tau) = \tau^{-\gamma}$.

It is shown in [8] that stabilization in problem (8) and (11) takes place if the equation of state $p = p(\tau)$ satisfies conditions (12) and (17) and problem (14) is solvable with respect to τ_s. In addition, the estimates

$$\begin{gathered} ||u, u_x, \tau_x | L_\infty(R^+; L_2(\Omega))|| + ||(\tau - \tau_s), (\tau - \tau_s)_x, u_{xx} | L_2(R^+; L_2(\Omega))|| \leqslant c, \\ ||\tau, \tau^{-1} | L_\infty(Q)|| \leqslant c, \quad Q = \{x, t : 0 < x < 1, \quad t > 0\} \end{gathered} \tag{18}$$

are valid for the solution. Here and below c denotes, generally speaking, various positive constants that are independent of T. Sometimes such constants will be denoted by subscripts. It is shown in the same paper that the convergence $||\tau - \tau_s, u | L_2(\Omega)|| \to 0$ is exponential for $t \to \infty$.

Temperature Asymptotic. Let us consider the special case of problem (6)–(10) for $V(t) = \text{const} \equiv V$ and $W(t) \equiv 0$. This means that the upper plane moves uniformly at constant velocity V along the y axis. For simplicity, we set $w \equiv 0$. Allowance for motion along the z axis would not complicate the problem mathematically. This is explained by the fact that further investigation consists in deriving time-independent estimates for flow parameters. The functions v and w appear in system (8)–(10) in the same way, and each of them can be found separately.

Denote $\alpha = v - V \int_0^x \tau(\xi, t) d\xi$. Then, $\alpha = 0$ for $x \in \partial\Omega$, and the equation $\alpha_t = \lambda(\rho\alpha_x)_x - uV$ is satisfied.

Here and below the norm and the scalar product in $L_2(\Omega)$ are denoted by $||\cdot||$ and $(\cdot,\cdot)$. From the energetic equality

$$\frac{1}{2}\frac{d}{dt}||\alpha||^2 + \lambda||\rho^{1/2}\alpha_x||^2 = -V(u, \alpha)$$

and the inequality $||\alpha|| \leqslant ||\rho^{1/2}\alpha_x||$ we find the estimate

$$||\alpha(t)||^2 \leqslant \exp(-\lambda t)||\alpha(0)||^2 + V^2 \int_0^t \exp(-\lambda(t-s))||u(s)||^2 ds. \tag{19}$$

Thus, by virtue of $\rho \geqslant c > 0$ and $||u | L_2(Q)|| \leqslant c$, the estimate

$$\int_0^\infty \left|\frac{d}{dt}||\alpha(t)||^2\right| dt + ||\alpha, \alpha_x | L_2(Q)|| + ||\alpha | L_\infty(R^+; L_2(\Omega))|| \leqslant c$$

is valid, which means, in particular, that $||\alpha(t)|| \to 0$ as $t \to \infty$. Moreover, it follows from (19) that this convergence is exponential, since this type of decrease with time is valid for the norm $||u(t)||$.

Multiplying Eq. (18) by α_{xx} and integrating over x, we obtain the equality

$$\frac{1}{2}\frac{d}{dt}||\alpha_x||^2 + \lambda||\rho^{1/2}\alpha_{xx}||^2 = V(u, \alpha_{xx}) - \lambda(\rho_x\alpha_x, \alpha_{xx}).$$

By virtue of estimates (18), this leads to the inequality

$$\sigma' \leqslant a(t)\sigma + b(t) \qquad (\sigma(t) = ||\alpha_x(t)||^2, \quad a(t) = c||\tau_x(t)||^2, \quad b(t) = c||u(t)||^2),$$

from which, according to the Gronwall lemma [5], we easily conclude that

$$||\alpha_x | L_\infty(R^+; L_2(\Omega))|| + ||\alpha_{xx} | L_2(Q)|| \leqslant c.$$

The estimates for functions τ, u, and α given above allow us to study the behavior of the function θ for $t \to \infty$, which is a solution of the problem

$$c_V\theta_t = æ(\rho\theta_x)_x + F, \qquad \theta_x\Big|_{\partial\Omega} = 0, \qquad \theta\Big|_{t=0} = \theta_0(x),$$

where $F = f + \lambda V^2\tau$, $f = -pu_x + \nu\rho u_x^2 + \lambda\rho\alpha_x^2 + 2V\lambda\alpha_x$.

We examine the function

$$\beta(x) = a\int_0^1\int_y^x \xi\tau_s(\xi)d\xi dy - \frac{a}{2}\Big(\int_0^x \tau_s(\xi)d\xi\Big)^2 + \frac{a}{2}\int_0^1\Big(\int_0^y \tau_s(\xi)d\xi\Big)^2 dy, \quad a = \frac{\lambda V^2}{æ}.$$

Clearly, it is a solution of the problem

$$æ(\rho_s\beta_x)_x + \lambda V^2(\tau_s - 1) = 0, \quad \beta_x\Big|_{\partial\Omega} = 0, \quad \int_0^1 \beta(x)dx = 0.$$

We consider the function $\psi(t)$:

$$c_V\psi(t) = \lambda V^2 t + A(t) \qquad \Big(A(t) = \int_0^t\int_0^1 f dx ds + c_V\int_0^1 \theta_0 dx\Big).$$

Then, the function $\zeta = \theta - \psi - \beta$ is a solution of the problem

$$c_V\zeta_t = æ(\rho\zeta_x)_x + æ((\rho - \rho_s)\beta_x)_x + G, \quad \zeta_x\Big|_{\partial\Omega} = 0, \quad \int_0^1 \zeta dx = 0, \quad \zeta\Big|_{t=0} = \theta_0 - \psi(0) - \beta(x). \qquad (20)$$

Here $G = f - \int_0^1 f dx + \lambda V^2(\tau - \tau_s)$.

The previous estimates guarantee the inclusion $G \in L_2(Q)$. To prove this, it will suffice to show that $u_x^2 \in L_2(Q)$.

From the inequality $J \equiv \max_x |u_x| \leqslant ||u_{xx}||$ we have

$$\int_Q u_x^4 dx dt \leqslant \int_0^\infty J(t)^2\Big(\int_\Omega u_x^2 dx\Big)dt \leqslant ||u_x|L_\infty\Big(R^+; L_2(\Omega)\Big)||^2||u_{xx}|L_2(Q)||^2,$$

which, by virtue of estimates (18), ensures the desired inclusion.

Multiplying Eq. (20) by ζ leads to the estimate

$$\int_0^\infty (||\zeta(t)||^2 + \Big|\frac{d}{dt}||\zeta(t)||^2\Big|)dt \leqslant c.$$

This means that $||\zeta(t)|| \to 0$ as $t \to \infty$.

Since the equality

$$\int_0^t\int_0^1 -pu_x dx ds = \int_0^1 \Phi(\tau)dx - \int_0^1 \Phi(\tau_0)dx$$

holds true, the function $A(t)$ is bounded uniformly over $t \in R^+$. In addition, by the Lebesgue theorem [9], the limit of the function A exists and is bounded for $t \to \infty$.

We denote $E = (1/c_V)\lim_{t\to\infty} A(t)$. Thus, this is proof that the temperature θ has the asymptotic behavior $\theta \to (\lambda/c_V)V^2 t + E + \beta(x)$ as $t \to \infty$ in the norm $L_2(\Omega)$.

It should be noted that the constant E depends not only on the initial temperature but on the entire history of motion, i.e., ultimately, on the initial state of the medium.

Remark 2. Let us explain why rotation of the liquid planes about vertical axes is impossible. If this were the case, it would mean that the equations

$$\rho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) = \text{ div } \mathbf{P} - \rho g, \quad \mathbf{P} = (-p + \lambda \text{ div } \mathbf{v})\mathbf{I} + 2\mu \mathbf{D}, \quad 2\mathbf{D} = \mathbf{v}_{i,j} + \mathbf{v}_{j,i}, \quad p = p(\rho)$$

are satisfied by the functions $\rho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t)$ of the form

$$\rho = \rho(x, t), \quad \mathbf{v} = \mathbf{v}_*(x, t) + \mathbf{e} \times \mathbf{r}\omega(x, t),$$

where $\mathbf{x} = (x, y, z)$; $\mathbf{r} = (0, y, z)$; $\mathbf{e} = (1, 0, 0)$; and $\omega(x_0, t)$ is the instantaneous angular velocity of the layer $x = x_0$. But this is true only in the case $\omega \equiv 0$.

Remark 3. If system (1)–(5) is completed by the equations of state $p = R\rho\theta$ and $U = c_V\theta$, it becomes a model of a perfec. polytropic viscous gas. This model is more complicated mathematically, since the velocity and density cannot be found independently of the temperature; the system becomes entirely indecomposable.

This work was supported by the Russian Foundation for Fundamental Research and International Scientific Foundation (Grant NR 5300), and also by the Grant Center of Saint Petersburg University (Grant ZN 257-94).

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